

Optimal geometric barriers for observability of heat semigroups on metric measure spaces

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The heat equation and observability

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded connected domain, $\omega \subset \Omega$ a nonempty open subset, and $T > 0$

Consider the heat equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } (0, T) \times \Omega \\ y = 0 & \text{on } (0, T) \times \partial\Omega \\ y(0, \cdot) = f & \text{in } \Omega \end{cases}$$

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Observability inequality there exists $C_T > 0$ such that

$$\forall f \in L^2(\Omega), \quad \|y(T, \cdot)\|_{L^2(\Omega)}^2 \leq C_T \int_0^T \|y(t, \cdot)\|_{L^2(\omega)}^2 dt \quad (1)$$

“Knowing y on the small region ω controls the total energy at time T ”

Why observability matters

Duality with control

Inequality (1) is equivalent to the **null-controllability** of the heat equation: for every $y_0 \in L^2(\Omega)$, there exists $u \in L^2((0, T) \times \omega)$ such that the solution of

$$\partial_t y - \Delta y = \mathbf{1}_\omega u, \quad y(0, \cdot) = y_0$$

satisfies $y(T, \cdot) = 0$

Lions, SIAM Rev. 1988 (Hilbert Uniqueness Method)

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Classical result

Inequality (1) holds for every $T > 0$ and every ω , with cost

$$C_T \leq C e^{K/T} \quad \text{as } T \rightarrow 0 \quad (2)$$

We define **the fast-control cost** as $K := \limsup_{T \rightarrow 0} T \ln C_T$

Lebeau–Robbiano, CPDE 1995; Fursikov–Imanuvilov, 1996

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The **small-time scale** $e^{K/T}$ is the signature of parabolic observability
 $\hookrightarrow$ Where does it come from?

Integrated observability inequalities

Two other formulations play a central role

(i) Finite-time integrated $\exists \tilde{\gamma}, \tilde{C} > 0, \forall T > 0$

$$\int_0^T e^{-\tilde{\gamma}/t} \|y(t)\|_{L^2(\Omega)}^2 dt \leq \tilde{C} \int_0^T \|y(t)\|_{L^2(\omega)}^2 dt$$

(ii) Infinite-time integrated $\exists \tilde{\gamma}_\infty, C_\infty > 0$

$$\int_0^{+\infty} e^{-\tilde{\gamma}_\infty/t} \|y(t)\|_{L^2(\Omega)}^2 dt \leq C_\infty \int_0^{+\infty} \|y(t)\|_{L^2(\omega)}^2 dt$$

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We denote by γ and γ_∞ the infima of the admissible exponents in (i) and (ii), respectively

Key facts:

(i) $\Leftrightarrow$ (1) for all $T > 0$, with $\gamma = K$

(i) $\Rightarrow$ (ii) with $\gamma \geq \gamma_\infty$

Geometric quantities and known bounds

Two natural **geometric quantities** measuring how “hidden” $\Omega \setminus \omega$ is

$$\mathcal{D}(\omega) = \sup\{\rho > 0 \mid \exists x_0, B(x_0, \rho) \subset \Omega \setminus \omega\} \quad (\text{largest hidden ball})$$

$$\mathcal{L}(\omega) = \sup_{x \in \Omega} d(x, \omega) \quad (\text{farthest point})$$

Always $\mathcal{L}(\omega) \geq \mathcal{D}(\omega)$

For measurable ω below, $d(x, \omega)$ is replaced by $d_{\text{ess}}(x, \omega) := \sup\{r > 0 : \mu(B(x, r) \cap \omega) = 0\}$

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Known lower bounds on the exponential cost

γ	γ_∞
$\geq \mathcal{D}(\omega)^2/2$	$\geq \mathcal{D}(\omega)^2/2$
$\geq \mathcal{L}(\omega)^2/2$	?

Zuazua (2001): $\mathcal{D}(\omega)^2/2$; Miller, JDE 2004: $\gamma \geq \mathcal{L}(\omega)^2/2$

The Ervedoza–Zuazua open problem

The maximal-distance bound is known for the finite-time constant:

$$\gamma \geq \frac{1}{2} \mathcal{L}(\omega)^2$$

Open problem

Does the infinite-time integrated constant satisfy

$$\gamma_\infty \geq \frac{1}{2} \mathcal{L}(\omega)^2 ?$$

Ervedoza–Zuazua, Arch. Ration. Mech. Anal. 202 (2011), Section 5 (“Lower bound for γ ”; their γ is our γ_∞)

The question we ask

Central question

Is the small-time scale $e^{-\gamma/t}$ **forced by the geometry**, or is it a by-product of the Carleman method used to prove observability?

We want to study **weighted observability inequalities**

$$\int_0^T h(t) \|y(t)\|_{L^2(\Omega)}^2 dt \leq \int_0^T \|y(t)\|_{L^2(\omega)}^2 dt$$

with *arbitrary* weight $h(t) \geq 0$, locally nondecreasing at 0

More precisely: h is bounded, continuous, nonnegative, and nondecreasing near 0

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More precisely: h is bounded, continuous, nonnegative, and nondecreasing near 0

Goal identify the **minimal admissible decay** of h at 0 and relate it to the **geometry of ω** in a **general geometric setting**

Abstract framework

We work on (X, d, μ) , a locally compact separable metric measure space, with μ Radon, $\text{supp} \mu = X$, $0 < \mu(B(x, r)) < +\infty$, and the **doubling property**: $\exists C > 0$,

$$\forall x \in X, \forall r > 0, \quad \mu(B(x, 2r)) \leq C \mu(B(x, r))$$

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We consider a **nonnegative self-adjoint operator** A on $L^2(X)$

We impose **three structural hypotheses** on A

The three hypotheses

(A1) Quantitative ultracontractivity For every $t > 0$, $e^{-tA} : L^2(X) \rightarrow L^\infty(X)$ is bounded, and for some $\alpha_1, C > 0$,

$$\forall t \in (0, 1], \quad \|e^{-tA}\|_{2 \rightarrow \infty} \leq C t^{-\alpha_1/2}$$

Equivalent to the existence of a bounded heat kernel $e^{-tA}f(x) = \int_X p_t(x, y)f(y)d\mu(y)$,
and $p_t(x, x) \leq C' t^{-\alpha_1}$

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(A2) Davies–Gaffney For all measurable $E, F \subset X$ and $t > 0$

$$\|\mathbf{1}_F e^{-tA} \mathbf{1}_E\|_{L^2 \rightarrow L^2} \leq e^{-d(E, F)^2/(4t)}$$

$\Leftrightarrow$ finite speed of propagation for the wave equation (Sikora, Math. Z. 2004)

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(A3) Pointwise local Weyl law For some $\alpha_2 > 0$, slowly varying χ_2 , and measurable $c : X \rightarrow (0, +\infty)$, for a.e. x ,

$$E_x(\lambda) := K_{\mathbf{1}_{[0, \lambda]}(A)}(x, x) \underset{\lambda \rightarrow +\infty}{\sim} c(x) \lambda^{\alpha_2} \chi_2(\lambda)$$

Hörmander, Acta Math. 1968, for elliptic operators ($\alpha_2 = d/2$)

Main result

Theorem (B., Hayat, Trélat 2026)

Let $T \in (0, +\infty]$, $\omega \subset X$ measurable with $\mathcal{L}(\omega) < +\infty$, and h an admissible weight such that

$$\int_0^T h(t) \|e^{-tA} f\|_{L^2(X)}^2 dt \leq \int_0^T \|e^{-tA} f\|_{L^2(\omega)}^2 dt$$

Then for every $0 < \kappa < \frac{1}{2}$, there exists $A_{T,\kappa} > 0$ such that

$$\forall t \in (0, T), \quad h(t) \leq A_{T,\kappa} \exp\left(-\kappa \frac{\mathcal{L}(\omega)^2}{t}\right)$$

Equivalently,

$$\limsup_{t \downarrow 0} t \log h(t) \leq -\frac{1}{2} \mathcal{L}(\omega)^2$$

Corollary: solving the open problem

Corollary (Infinite-time constant lower bound)

If for some $\gamma_\infty > 0$ and $C_\infty > 0$

$$\int_0^{+\infty} e^{-\gamma_\infty/t} \|e^{-tA}f\|_{L^2(X)}^2 dt \leq C_\infty \int_0^{+\infty} \|e^{-tA}f\|_{L^2(\omega)}^2 dt$$

for $f \in (\ker A)^\perp$ for which the right-hand side is finite, then necessarily

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This is exactly the maximal-distance lower bound asked in the 2011 open problem.

Ervedoza–Zuazua, Arch. Ration. Mech. Anal. 202 (2011), Section 5

Why the coefficient $\frac{1}{2}$ is sharp

Universal lower-bound threshold

Our corollary gives

$$\gamma_\infty \geq \frac{1}{2} \mathcal{L}(\omega)^2$$

In one dimension, and in multidimensional radial configurations where $\Omega \setminus \omega$ is a ball, the optimal exponential threshold is exactly

$$\frac{1}{2} \mathcal{L}(\omega)^2.$$

Hence **no larger universal coefficient can hold**

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Sharpness concerns the logarithmic/exponential threshold, the endpoint estimate with a bounded prefactor is a stronger statement

Where the framework applies

Setting	α_1	α_2	χ_2	$c(x)$
Compact Riemannian, $-\Delta_g + V$	$d/2$	$d/2$	1	$\frac{\omega_d}{(2\pi)^d}$
Schrödinger on $\mathbb{R}^d$, $-\Delta + V$	$d/2$	$d/2$	1	$\frac{\omega_d}{(2\pi)^d}$
Grushin sub-Laplacian	$3/2$	1	1	$c_{sR}(x)$
δ' -coupled Laplacians on metric graphs	$1/2$	$1/2$	1	$1/\pi$

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$\Rightarrow$ Theorem gives the lower bound $\gamma_\infty \geq \frac{1}{2} \mathcal{L}(\omega)^2$ in all these settings

The framework is robust enough to include settings with
non-continuous heat kernels (graphs with δ' -coupling)

A geometric example: the Grushin operator

On $(-1, 1) \times (0, 1)$, consider the degenerate operator

$$G = -\partial_x^2 - x^2 \partial_y^2$$

Subelliptic, with sub-Riemannian distance singular at $\{x = 0\}$

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Known phenomenon

Observability from a vertical strip $\omega = (a, b) \times (0, 1)$,

$0 < a < b < 1$, *has a positive minimal time*

Distinctively non-Riemannian behavior

Beauchard–Cannarsa–Guglielmi, JEMS 2014.

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Consequence of our theorem for every infinite-time integrated observability inequality,

$$\gamma_\infty \geq \frac{1}{2} \mathcal{L}_{sR}(\omega)^2$$

where $\mathcal{L}_{sR}$ is computed with the *sub-Riemannian* (Grushin) distance

Proof strategy on $\mathbb{R}^d$ (with $A = -\Delta$)

Choose $x \notin \overline{\omega}$, set $\delta = d(x, \omega)$, and fix $0 < r < \delta$.

Sharp Gaussian wave packet

$$g_{r,t,x} := (\cosh(r\sqrt{-\Delta}) - 1)p_t(\cdot, x), \quad p_t = e^{t\Delta}\delta_x$$

In Fourier variables,

$$\widehat{g}_{r,t,x}(\xi) = e^{-ix \cdot \xi} (\cosh(r|\xi|) - 1) e^{-t|\xi|^2}$$

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The growing branch is a Gaussian centered at the sharp frequency $|\xi| = r/(2t)$:

$$r|\xi| - t|\xi|^2 = \frac{r^2}{4t}$$

After squaring, its natural size is $e^{r^2/(2t)}$

The packet: two competing estimates

Lower bound on $\mathbb{R}^d$ By Plancherel, for every $\eta > 0$ and small t ,

$$\|g_{r,t,x}\|_{L^2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (\cosh(r|\xi|) - 1)^2 e^{-2t|\xi|^2} d\xi \gtrsim e^{(1-\eta)r^2/(2t)}$$

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Upper bound on ω The packet has the wave-to-heat representation

Transmutation formula for the packet, $w(x, y)$ is the wave kernel, g_t is the Euclidean heat kernel

$$g_{r,t,x}(y) = \langle w(x, y), \Phi_{r,t} - g_t \rangle, \quad \Phi_{r,t}(s) = \frac{e^{(r^2-s^2)/(4t)}}{\sqrt{4\pi t}} \cos\left(\frac{rs}{2t}\right)$$

Finite propagation forces $|s| \geq d(x, y) \geq \delta$ on ω , hence

$$\|g_{r,t,x}\|_{L^2(\omega)}^2 \lesssim t^{-N} e^{-(\delta^2-r^2)/(2t)}$$

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Apply observability to a regularized packet:

$$h(\tau) \leq Ae^{-\kappa r^2/\tau} \quad (\kappa < \tfrac{1}{2})$$

then let $r \uparrow d(x, \omega)$ and optimize in x

From $\mathbb{R}^d$ to a general operator A

The same packet is defined spectrally:

where $K_{f(A)}(x, y)$ is the integral kernel of $f(A)$

$$\tilde{g}_{r,t,x} = K_{(\cosh(r\sqrt{A})-1)e^{-tA}}(\cdot, x).$$

(A1) Pointwise spectral measure replaces Fourier analysis:

$$\|\tilde{g}_{r,t,x}\|_{L^2(X)}^2 = \int_0^{+\infty} (\cosh(r\sqrt{\lambda}) - 1)^2 e^{-2t\lambda} d\nu_x(\lambda)$$

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(A2) Weak wave kernel + Kannai transmutation gives finite propagation and the upper bound on ω

Kannai, CPDE 1977. The construction works without compact resolvent or heat-kernel continuity

Summary

Main message

The small-time scale $e^{-\gamma/t}$ in integrated observability is **intrinsic and geometric**, not an artefact of Carleman estimates

Sharp universal lower bound $\gamma_\infty \geq \frac{1}{2} \mathcal{L}(\omega)^2$

This resolves the maximal-distance question for the infinite-time constant

Ervedoza–Zuazua, Arch. Ration. Mech. Anal. 202 (2011), Section 5

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What was hidden in this talk

Everything works for operators on **sections of a Euclidean vector bundle** $\mathcal{V} \rightarrow X$ (connection Laplacians, coupled heat systems, Hodge Laplacian on k -forms)

Applies to non-continuous heat kernels (e.g. δ' -coupling on metric graphs)