

Geometric barriers for observability of heat semigroups on metric measure spaces

Vincent Boulard

Bézout Day 2026

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Wednesday, May 27, 2026

Joint work with Amaury Hayat (CERMICS) & Emmanuel Trélat (LJLL)

The heat equation and observability

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded connected domain, $\omega \subset \Omega$ a nonempty open subset, and $T > 0$

Consider the heat equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } (0, T) \times \Omega \\ y = 0 & \text{on } (0, T) \times \partial\Omega \\ y(0, \cdot) = f & \text{in } \Omega \end{cases}$$

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Observability inequality there exists $C_T > 0$ such that

$$\forall f \in L^2(\Omega), \quad \|y(T, \cdot)\|_{L^2(\Omega)}^2 \leq C_T \int_0^T \|y(t, \cdot)\|_{L^2(\omega)}^2 dt \quad (1)$$

“Knowing y on the small region ω controls the total energy at time T ”

Why observability matters

Duality with control

Inequality (1) is equivalent to the **null-controllability** of the heat equation: for every $y_0 \in L^2(\Omega)$, there exists $u \in L^2((0, T) \times \omega)$ such that the solution of

$$\partial_t y - \Delta y = \mathbf{1}_\omega u, \quad y(0, \cdot) = y_0$$

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Classical result

Inequality (1) holds for every $T > 0$ and every ω , with cost

$$C_T \leq C e^{K/T} \quad \text{as } T \rightarrow 0 \quad (2)$$

We define **the fast-control cost** as $K := \limsup_{T \rightarrow 0} T \ln C_T$

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The **small-time scale** $e^{K/T}$ is the signature of parabolic observability

$\hookrightarrow$ Where does it come from?

Integrated observability inequalities

Two other formulations play a central role

(i) Finite-time integrated $\exists \gamma, \tilde{C} > 0, \forall T > 0$

$$\int_0^T e^{-\gamma/t} \|y(t)\|_{L^2(\Omega)}^2 dt \leq \tilde{C} \int_0^T \|y(t)\|_{L^2(\omega)}^2 dt$$

(ii) Infinite-time integrated $\exists \gamma_\infty, C_\infty > 0$

$$\int_0^{+\infty} e^{-\gamma_\infty/t} \|y(t)\|_{L^2(\Omega)}^2 dt \leq C_\infty \int_0^{+\infty} \|y(t)\|_{L^2(\omega)}^2 dt$$

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Key facts:

(i) $\Leftrightarrow$ (1) for all $T > 0$, with $\gamma = K$

(i) $\Rightarrow$ (ii) with $\gamma \geq \gamma_\infty$

Geometric quantities and known bounds

Two natural **geometric quantities** measuring how “hidden” $\Omega \setminus \omega$ is

$$\mathcal{D}(\omega) = \sup\{\rho > 0 \mid \exists x_0, B(x_0, \rho) \subset \Omega \setminus \omega\} \quad (\text{largest hidden ball})$$

$$\mathcal{L}(\omega) = \sup_{x \in \Omega} d(x, \omega) \quad (\text{farthest point})$$

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Known lower bounds on the exponential cost

γ	γ_∞
$\geq \mathcal{D}(\omega)^2/2$	$\geq \mathcal{D}(\omega)^2/2$
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Open problem

Can we have a similar lower bound between γ_∞ and $\mathcal{L}(\omega)^2$?

The question we ask

Central question

Is the small-time scale $e^{-\gamma/t}$ **forced by the geometry**, or is it a by-product of the Carleman method used to prove observability?

We want to study **weighted observability inequalities**

$$\int_0^T h(t) \|y(t)\|_{L^2(\Omega)}^2 dt \leq \int_0^T \|y(t)\|_{L^2(\omega)}^2 dt$$

with *arbitrary* weight $h(t) \geq 0$, locally nondecreasing at 0

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Goal identify the **minimal admissible decay** of h at 0 and relate it to the **geometry of ω** in a **general geometric setting**

Abstract framework

We work on (X, d, μ) , a locally compact separable metric measure space, with μ Radon, $\text{supp}\mu = X$, and the **doubling property**: $\exists C > 0$,

$$\forall x \in X, \forall r > 0, \quad \mu(B(x, 2r)) \leq C \mu(B(x, r))$$

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We consider a **nonnegative self-adjoint operator** A on $L^2(X)$

We impose **three structural hypotheses** on A

The three hypotheses

(A1) Ultracontractivity For all $t > 0$

$e^{-tA} : L^2(X) \rightarrow L^\infty(X)$ is bounded.

Equivalent to the existence of a bounded heat kernel

$$e^{-tA}f(x) = \int_X p_t(x, y)f(y)d\mu(y)$$

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(A2) Davies–Gaffney For all measurable $E, F \subset X$ and $t > 0$

$$\|\mathbf{1}_F e^{-tA} \mathbf{1}_E\|_{L^2 \rightarrow L^2} \leq e^{-d(E, F)^2/(4t)}$$

$\Leftrightarrow$ finite speed of propagation for the wave equation

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(A3) Local Weyl law $\exists \alpha > 0$, χ slowly varying, $c \in L^\infty(X)$

$$K_{\mathbf{1}_{[0,\lambda]}(A)}(x, x) \underset{\lambda \rightarrow +\infty}{\sim} c(x) \lambda^\alpha \chi(\lambda)$$

Classical Hörmander asymptotics for elliptic operators ($\alpha = d/2$)

Main result

Theorem (B., Hayat, Trélat 2026)

Let $T \in (0, +\infty]$, $\omega \subset X$ measurable with $\mathcal{L}(\omega) \in (0, +\infty)$, and h a weight giving a weighted observability inequality

$$\int_0^T h(t) \|e^{-tA} f\|_{L^2(X)}^2 dt \leq \int_0^T \|e^{-tA} f\|_{L^2(\omega)}^2 dt$$

Then for every $\kappa < \frac{1}{4}$, there exists $A_T > 0$ such that

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Interpretation

The exponential scale $e^{-c/t}$ is **not optional**, it is forced by the integrated observability inequality itself

The geometric constant $\mathcal{L}(\omega)^2$ appears **intrinsically**

Corollary: solving the open problem

Corollary (Infinite-time constant lower bound)

If for some $\gamma_\infty > 0$ and $C_\infty > 0$

$$\int_0^{+\infty} e^{-\gamma_\infty/t} \|e^{-tA}f\|_{L^2(X)}^2 dt \leq C_\infty \int_0^{+\infty} \|e^{-tA}f\|_{L^2(\omega)}^2 dt$$

then necessarily

$$\gamma_\infty \geq \frac{1}{4} \mathcal{L}(\omega)^2$$

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Comments

Answers positively the open problem of (with constant $\frac{1}{4}$ instead of the expected $\frac{1}{2}$)

Where the framework applies

Setting	α	χ
Compact Riemannian, $-\Delta_g + V$	$d/2$	cst
Schrödinger on $\mathbb{R}^d$, $-\Delta + V$, $V \geq 0$	$d/2$	cst
Sub-Riemannian sub-Laplacian	α_{sR}	log corr.
δ' -coupled Laplacians on metric graphs	$1/2$	$1/\pi$

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$\Rightarrow$ Theorem gives the lower bound $\gamma_\infty \geq \frac{1}{4} \mathcal{L}(\omega)^2$ in all these settings.

The framework is robust enough to include settings with
non-continuous heat kernels (graphs with δ' -coupling)

A geometric example: the Grushin operator

On $(-1, 1) \times (0, 1)$, consider the degenerate operator

$$G = -\partial_x^2 - x^2 \partial_y^2$$

Subelliptic, with sub-Riemannian distance singular at $\{x = 0\}$

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Observability from a vertical strip $\omega \supset (0, b) \times (0, 1)$ *requires a positive minimal time*

Distinctively non-Riemannian behavior

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Distinctively non-Riemannian behavior

Consequence of our theorem when observability holds for all $T > 0$ (e.g. ω touches $\{x = 0\}$)

$$\gamma_\infty \geq \frac{1}{4} \mathcal{L}_{sR}(\omega)^2$$

where $\mathcal{L}_{sR}$ is computed with the *sub-Riemannian* (Grushin) distance

$\Rightarrow$ New lower bound on the fast-control cost in a sub-Riemannian setting

Proof strategy on $\mathbb{R}^d$ (with $A = -\Delta$)

Test functions For $x \notin \bar{\omega}$ and $k \geq 0$, plug

$$f = \Delta^k \delta_x \in \mathcal{E}'(\mathbb{R}^d)$$

into the weighted observability inequality. The semigroup $e^{t\Delta}$ smoothes it

$$(e^{t\Delta} f)(y) = \Delta^k p_t(y, x), \quad p_t(y, x) = (4\pi t)^{-d/2} e^{-|x-y|^2/(4t)}$$

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Two estimates needed

(L) Lower bound for $\|\Delta^k p_t(\cdot, x)\|_{L^2(\mathbb{R}^d)}$: full L^2 -norm

(U) Upper bound for $\|\Delta^k p_t(\cdot, x)\|_{L^2(\omega)}$: localized away from x

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The ratio **(L)/(U)**, summed in k with appropriate weights, will produce the exponential constraint on h

The key computation

Lower bound on $\mathbb{R}^d$ By Plancherel

$$\|\Delta^k p_t(\cdot, x)\|_{L^2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{4k} e^{-2t|\xi|^2} d\xi = c_d \frac{\Gamma(2k + d/2)}{(2t)^{2k+d/2}}$$

if $a < d(x, \omega)^2/2$, taking $\sup_{x \notin \bar{\omega}}$ saturates the bound at $\mathcal{L}(\omega)^2$

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Upper bound on ω Since $d(x, \omega) \geq \delta > 0$, the **Gaussian decay** of p_t controls $\|\Delta^k p_t(\cdot, x)\|_{L^2(\omega)}^2$ uniformly in k and t

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Sum the inequality Dividing by suitable factorials and multiplying by a^{2k} and summing in k gives

$$\int_0^T h(t) \cosh\left(\frac{a}{2t}\right) \frac{dt}{t^{d/2}} < +\infty$$

if $a < d(x, \omega)^2/2$, taking $\sup_{x \notin \bar{\omega}}$ saturates the bound at $\mathcal{L}(\omega)^2$

The two abstract tools (general A)

In the general setting (X, d, μ, A) , the Fourier transform is not available, we replace it by

Kannai transmutation formula (relating heat and wave kernels)

$$e^{-tA} = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-s^2/(4t)} \cos(s\sqrt{A}) ds$$

Combined with (A2): *the wave kernel $w(x, y)$ vanishes for $|s| < d(x, y)$*

$\Rightarrow$ pointwise upper bounds on $A^k p_t(x, y)$

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Local Weyl law (A3) + pointwise spectral measures

$$\int_X |A^k p_t(\cdot, x)|^2 d\mu = \int_0^{+\infty} \lambda^{2k} e^{-2t\lambda} d\nu_x(\lambda)$$

where ν_x is the pointwise spectral measure at x

$\Rightarrow$ lower bounds via Weyl asymptotics

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$\Rightarrow$ lower bounds via Weyl asymptotics

The construction of ν_x without compact resolvent or kernel continuity is a key technical contribution

Summary

Main message

The small-time scale $e^{-\gamma/t}$ in integrated observability is **intrinsic and geometric**, not an artefact of Carleman estimates

New lower bound $\gamma_\infty \geq \frac{1}{4} \mathcal{L}(\omega)^2$

Summary

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New lower bound $\gamma_\infty \geq \frac{1}{4} \mathcal{L}(\omega)^2$

What was hidden in this talk

Everything works for operators on **sections of a Euclidean vector bundle** $\mathcal{V} \rightarrow X$ (connection Laplacians, coupled heat systems, Hodge Laplacian on k -forms)

Applies to non-continuous heat kernels (e.g. δ' -coupling on metric graphs)