

Nonlinear semigroups, LaSalle invariance principle, and asymptotics of the Allen–Cahn equation

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Abstract

These notes accompany a one-hour lecture given at CERMICS. The lecture is organized in two parts. The first develops a general framework: nonlinear semigroups on metric spaces, Lyapunov stability, and LaSalle’s invariance principle. The second part applies this framework to a concrete nonlinear PDE—the Chafee–Infante / Allen–Cahn equation

$$\partial_t u = \Delta u + u - u^3$$

—for which we construct the semigroup by a fixed point argument, identify the asymptotic behaviour of trajectories via LaSalle, and ultimately prove *convergence to a single equilibrium* using the Łojasiewicz–Simon inequality. The interplay between the two parts illustrates how abstract dynamical arguments combine with PDE analysis to yield sharp asymptotic results.

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Introduction and guiding thread

Three model equations will accompany us throughout the lecture:

$$\dot{x} = -\nabla U(x), \quad \partial_t u = \Delta u, \quad \partial_t u = \Delta u + u - u^3. \quad (1)$$

The first is a gradient flow on $\mathbb{R}^d$; the second is the linear heat equation, which you have already encountered as a C_0 -semigroup on $L^2(\Omega)$; the third is the Allen–Cahn or Chafee–Infante equation (depending on the boundary conditions), a genuinely nonlinear semilinear parabolic PDE with multiple stationary solutions.

All three fit into the same abstract framework—a nonlinear semigroup on a metric space—and all three are gradient flows of an energy functional, respectively U , $\frac{1}{2}\|\nabla u\|_{L^2}^2$, and

$$E(u) = \int_{\Omega} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(u^2 - 1)^2 \right) dx.$$

The Chafee–Infante energy is the Allen–Cahn double-well functional: its two global minima at $u \equiv \pm 1$ (formally, ignoring the boundary condition) correspond to the two “phases” of a phase-transition problem.

Our ambition here is:

- (i) build the right abstract vocabulary (Part I);
- (ii) use it on the Chafee–Infante equation to construct the nonlinear semigroup, show energy dissipation, and conclude, via a sharp combination of LaSalle and Łojasiewicz–Simon, that *every trajectory converges in H_0^1 to a single stationary solution* (Part II).

The convergence statement in (ii) is non-trivial: for infinite-dimensional gradient flows of merely C^∞ functionals, trajectories can fail to converge. That analyticity of the energy rescues us is a deep fact due to L. Simon [12].

Part I

General framework

1 Nonlinear semigroups on metric spaces

1.1 Definition and first properties

In your earlier courses, you have seen that a broad class of linear evolution equations in a Banach space X ,

$$\partial_t u = Au, \quad u(0) = u_0,$$

with A an unbounded generator, gives rise to a C_0 -semigroup $(S_t)_{t \geq 0}$ of bounded linear operators on X , satisfying $S_0 = \text{Id}$, $S_{t+s} = S_t S_s$, and continuity in t . The Hille–Yosida theory gives a complete picture in the linear case; see [11].

For *nonlinear* equations, it is natural to work directly in a metric space, because the solution map is no longer an operator but just a (typically Lipschitz) transformation.

Definition 1.1 (Continuous nonlinear semigroup). Let (X, d) be a complete metric space. A *continuous nonlinear semigroup* on X is a family $(S_t)_{t \geq 0}$ of maps $S_t : X \rightarrow X$ such that:

$$(S1) \ S_0 = \text{Id};$$

$$(S2) \ S_{t+s} = S_t \circ S_s \text{ for all } t, s \geq 0;$$

$$(S3) \ (t, x) \mapsto S_t x \text{ is continuous from } [0, \infty) \times X \text{ to } X.$$

If, in addition, each S_t is nonexpansive, *i.e.* $d(S_t x, S_t y) \leq d(x, y)$ for all $x, y \in X$ and $t \geq 0$, we say that $(S_t)_{t \geq 0}$ is a *contraction semigroup*.

Remark 1.2 (No generator in general). Unlike the linear case, there is no straightforward notion of “generator” for a nonlinear semigroup on a metric space: defining $Ax := \lim_{t \downarrow 0} (S_t x - x)/t$ requires a vector-space structure, which we do not have. In Hilbert or Banach spaces, the correct substitute is the theory of m -accretive (or maximal monotone) operators, whose cornerstone is the Crandall–Liggett generation theorem [6]; a self-contained French exposition is [2]. In genuine metric spaces, one can still define gradient flows *variationally*, via curves of maximal slope; this is the subject of [1].

1.2 Examples

Example 1.3 (Finite-dimensional ODE flows). Let $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be globally Lipschitz. Cauchy–Lipschitz provides, for every $x_0 \in \mathbb{R}^d$, a unique global solution $x \in C^1(\mathbb{R}; \mathbb{R}^d)$ of $\dot{x} = b(x)$, $x(0) = x_0$. Setting $S_t x_0 := x(t)$ defines a continuous semigroup on $(\mathbb{R}^d, |\cdot|)$ —nonlinear as soon as b is.

Example 1.4 (Linear C_0 -semigroups). Every C_0 -semigroup $(S_t)_{t \geq 0}$ on a Banach space X is, in particular, a continuous semigroup in the sense of Definition 1.1. The prototype is the Dirichlet heat semigroup $(e^{t\Delta})_{t \geq 0}$ on $L^2(\Omega)$ with $\Omega \subset \mathbb{R}^N$ bounded smooth domain, which is a contraction semigroup in our nonlinear sense.

Example 1.5 (Guiding PDE; announcement). Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain ($N \in \{1, 2, 3\}$). Consider the Chafee–Infante equation with Dirichlet boundary (with Neumann it would be the Allen–Cahn equation) conditions,

$$\begin{cases} \partial_t u = \Delta u + u - u^3 & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ u(0, \cdot) = u_0. \end{cases} \quad (2)$$

We will show in Part II that (2) defines a continuous semigroup on the Hilbert space $V := H_0^1(\Omega)$. This semigroup is genuinely nonlinear; its rich phenomenology (multiple equilibria, double-well energy) is what makes it worth studying.

2 Lyapunov stability

From now on, (X, d) is a complete metric space, and $(S_t)_{t \geq 0}$ is a continuous semigroup on X .

2.1 Equilibria and stability

Definition 2.1 (Equilibrium). A point $x_\star \in X$ is an *equilibrium* of $(S_t)_{t \geq 0}$ if $S_t x_\star = x_\star$ for all $t \geq 0$.

Definition 2.2 (Stability). An equilibrium $x_\star$ is said to be

- *Lyapunov stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all x_0 with $d(x_0, x_\star) < \delta$, we have $d(S_t x_0, x_\star) < \varepsilon$ for all $t \geq 0$;
- *asymptotically stable* if it is stable and if for all x_0 , we have $d(S_t x_0, x_\star) \rightarrow 0$ as $t \rightarrow \infty$.

2.2 Lyapunov functions

The core idea of Lyapunov's method is that stability can be read off a scalar quantity that *decreases along trajectories*. For conservative systems (e.g. Schrödinger, wave), one has *conserved* quantities (energy); for dissipative systems, energy *dissipates*.

Definition 2.3 (Lyapunov function). A continuous function $V : X \rightarrow \mathbb{R}$ is a *Lyapunov function* for $(S_t)_{t \geq 0}$ at an equilibrium x_* if

(L1) There exists $c : [0, \infty) \rightarrow [0, \infty)$ continuous, increasing with $c(0) = 0$ such that $V(x) - V(x_*) \geq c(d(x, x_*))$ for all $x \in X$.

(L2) $t \mapsto V(S_t x)$ is nonincreasing for every $x \in X$.

Theorem 2.4 (Lyapunov's theorem). *If a Lyapunov function exists at x_* , then x_* is Lyapunov stable.*

Proof. Let $\varepsilon > 0$. Set $\eta := c(\varepsilon) > 0$ and define

$$\Omega_\eta := \{x \in X : V(x) < V(x_*) + \eta\}.$$

For $x \in \Omega_\eta$, (L1) gives $c(d(x, x_*)) \leq V(x) - V(x_*) < \eta = c(\varepsilon)$. Since c is strictly increasing, $d(x, x_*) < \varepsilon$. Hence $\Omega_\eta \subset B(x_*, \varepsilon)$. Let $x_0 \in \Omega_\eta$. By (L2), $t \mapsto V(S_t x_0)$ is nonincreasing on $[0, \infty)$, so

$$V(S_t x_0) \leq V(x_0) < V(x_*) + \eta \quad \forall t \geq 0,$$

i.e. $S_t x_0 \in \Omega_\eta$. By continuity of V at x_* and the strict inequality $V(x_*) < V(x_*) + \eta$, there exists $\delta > 0$ such that $d(x_0, x_*) < \delta$ implies $V(x_0) < V(x_*) + \eta$, i.e. $x_0 \in \Omega_\eta$. Hence, $S_t x_0 \in B(x_*, \varepsilon)$ for every $t \geq 0$. $\square$

Note that Lyapunov's theorem does not yield asymptotic stability, because the dissipation is not strict: i.e. we can have $V(S_t x_0) = V(x_0)$ for all $t \geq 0$. LaSalle's invariance principle is the tool designed to handle this situation.

2.3 Toy example: gradient flow in $\mathbb{R}^d$

Example 2.5 (Gradient flow). Let $U \in C^2(\mathbb{R}^d; \mathbb{R})$ and x_* a strict local minimum with $\nabla U(x_*) = 0$. The ODE $\dot{x} = -\nabla U(x)$ generates a semigroup as in Example 1.3, and $V := U$ is a Lyapunov function at x_* :

$$\frac{d}{dt}U(x(t)) = \nabla U(x(t)) \cdot \dot{x}(t) = -|\nabla U(x(t))|^2 \leq 0.$$

Note that V is *not* strict if x_* is non-isolated (continua of critical points), which is precisely the situation we will deal with in Part II. This is the conceptual reason why LaSalle is needed.

3 LaSalle's invariance principle

When $\dot{V} \leq 0$ is *not strict*—the typical case for gradient flows, since $\dot{V} = 0$ precisely at critical points—Lyapunov's theorem alone does not yield asymptotic stability. LaSalle's principle is the tool designed to handle exactly this situation: it converts a nonstrict dissipation into a description of where trajectories asymptotically end up.

3.1 LaSalle's theorem

We need a small piece of vocabulary. A subset $A \subset X$ is said to be *positively invariant* under $(S_t)_{t \geq 0}$ if $S_t(A) \subset A$ for every $t \geq 0$.

Theorem 3.1 (LaSalle's invariance principle). *Let $V : X \rightarrow \mathbb{R}$ be continuous, bounded below, and such that $t \mapsto V(S_t x)$ is nonincreasing for every $x \in X$. Let $x_0 \in X$ be such that the positive orbit $\gamma^+(x_0) := \{S_t x_0 : t \geq 0\}$ is relatively compact in X . Then*

$$d(S_t x_0, \mathcal{I}) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where $\mathcal{I}$ is the largest positively invariant subset of $\mathcal{E} := \{x \in X : V(S_t x) = V(x) \text{ for all } t \geq 0\}$.

Proof. Since $t \mapsto V(S_t x_0)$ is nonincreasing and bounded below, it converges to some limit $\ell \in \mathbb{R}$.

Suppose by contradiction that the conclusion fails: there exist $\varepsilon > 0$ and a sequence $t_n \rightarrow \infty$ such that

$$d(S_{t_n} x_0, \mathcal{I}) \geq \varepsilon \quad \forall n. \quad (3)$$

By relative compactness of $\gamma^+(x_0)$, we extract a subsequence (still denoted t_n) such that $S_{t_n} x_0 \rightarrow y$ for some $y \in X$. By continuity of V , $V(y) = \lim V(S_{t_n} x_0) = \ell$.

We now examine the future trajectory of y . For every $\tau \geq 0$, continuity of S_τ gives

$$S_\tau y = \lim_{n \rightarrow \infty} S_\tau(S_{t_n} x_0) = \lim_{n \rightarrow \infty} S_{t_n + \tau} x_0,$$

so $V(S_\tau y) = \lim V(S_{t_n + \tau} x_0) = \ell$ as well (the limit $\lim_{t \rightarrow \infty} V(S_t x_0) = \ell$ is unaffected by the additive shift τ inside $t_n + \tau \rightarrow \infty$). Hence $V(S_\tau y) = V(y)$ for every $\tau \geq 0$, which means $y \in \mathcal{E}$.

The same argument applied at any $\tau' \geq 0$ in place of 0 shows that $V(S_\tau(S_{\tau'} y)) = \ell$ for every $\tau, \tau' \geq 0$, so the entire forward orbit $\Gamma := \{S_\tau y : \tau \geq 0\}$ lies in $\mathcal{E}$. Now Γ is positively invariant by construction (since $S_t(S_\tau y) = S_{t+\tau} y \in \Gamma$). By maximality of $\mathcal{I}$, $\Gamma \subset \mathcal{I}$, and in particular $y \in \mathcal{I}$.

But then $d(S_{t_n} x_0, \mathcal{I}) \leq d(S_{t_n} x_0, y) \rightarrow 0$, contradicting (3). $\square$

Remark 3.2 (Reading the proof). The essence of the argument fits in one sentence: along the trajectory, V converges to some ℓ ; if the trajectory failed to approach $\mathcal{I}$, we would extract a limit point y whose entire forward orbit also has $V \equiv \ell$, hence lies in $\mathcal{E}$ and is positively invariant, hence in $\mathcal{I}$ —a contradiction. The relative compactness of the orbit is what allows us to extract the limit point y .

Example 3.3 (Damped pendulum). Consider a pendulum of unit mass and unit length subject to linear friction:

$$\ddot{\theta} + b \dot{\theta} + \sin \theta = 0, \quad b > 0.$$

Setting $x = (\theta, \omega)$ with $\omega := \dot{\theta}$, we get a system on $\mathbb{R}^2$:

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\omega} = -b\omega - \sin \theta. \end{cases} \quad (4)$$

The natural Lyapunov function is the total mechanical energy,

$$V(\theta, \omega) := \frac{1}{2} \omega^2 + (1 - \cos \theta).$$

Along any trajectory of (4),

$$\dot{V} = \omega \dot{\omega} + \sin \theta \dot{\theta} = \omega(-b\omega - \sin \theta) + \omega \sin \theta = -b\omega^2 \leq 0.$$

So V is a Lyapunov function at the equilibrium $x_\star = (0, 0)$: Theorem 2.4 yields *stability*. However, the dissipation is *not strict*: $\dot{V}$ vanishes not only at $x_\star$ but on the entire axis $\{\omega = 0\}$.

LaSalle nevertheless gives local *asymptotic stability* near x_* . The trajectories starting in a sublevel set of V remain bounded, hence relatively compact in $\mathbb{R}^2$. By Theorem 3.1, every trajectory approaches the largest positively invariant subset of $\{\dot{V} = 0\} = \{\omega = 0\}$. For a trajectory to remain on $\{\omega = 0\}$, we need $\omega(t) \equiv 0$, hence $\dot{\omega} \equiv 0$, hence $\sin \theta \equiv 0$, so $\theta \equiv k\pi$. The largest invariant set in $\{\omega = 0\}$ is therefore the discrete set of equilibria $\{(k\pi, 0) : k \in \mathbb{Z}\}$. Near x_* , the only such point is x_* itself, so every trajectory close enough to x_* converges to it.

Remark 3.4 (Where PDE analysis enters). The precompactness assumption is the nontrivial PDE-theoretic input: in finite dimensions, boundedness is enough; in infinite dimensions, one must exploit a smoothing or compactness effect. In Part II, this comes from parabolic smoothing combined with Rellich's compact embedding.

For a much more detailed treatment, including delicate versions for PDEs, see [3], [9], [7], or the classical monograph [10].

Part II

Application to the Allen–Cahn equation

We now apply the framework of Part I to the Allen–Cahn equation (2). The programme is:

- (i) build a nonlinear semigroup on $V = H_0^1(\Omega)$ by a Banach fixed point (Section 4);
- (ii) identify the energy as a Lyapunov functional and apply LaSalle to show that trajectories asymptotically approach the set of stationary solutions (Section 5);
- (iii) invoke the Łojasiewicz–Simon inequality to upgrade this to convergence to a single equilibrium (Section 6).

Setting. Let $\Omega \subset \mathbb{R}^N$ with $N \in \{1, 2, 3\}$ be bounded with smooth boundary. Let $A := -\Delta$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, acting on $L^2(\Omega)$. The operator A is self-adjoint, positive, and has compact resolvent. We work in $V := H_0^1(\Omega)$ with $\|v\|_V^2 := \int_\Omega |\nabla v|^2$; the Poincaré inequality $\|v\|_{L^2} \leq C_P \|v\|_V$ and the Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^6(\Omega), \quad \|v\|_{L^6} \leq C_S \|v\|_V,$$

are the main analytic tools; the constraint $N \leq 3$ is precisely there to ensure this embedding.

The linear heat semigroup $(e^{-tA})_{t \geq 0}$ is a C_0 -contraction semigroup on L^2 . It also acts contractively on V , and enjoys the smoothing estimate

$$\|e^{-tA}v\|_V \leq Ct^{-1/2}\|v\|_{L^2}, \quad t > 0. \tag{5}$$

We write $f(u) := u^3 - u$ for the nonlinearity and use the *double-well Ginzburg–Landau energy*

$$E(u) := \int_\Omega \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{4} (u^2 - 1)^2 \right) dx, \quad u \in V. \tag{6}$$

Note that $E \geq 0$ on V , which is one reason for choosing this form; see Remark 3.5. The potential $W(u) := \frac{1}{4}(u^2 - 1)^2$ satisfies $W'(u) = u^3 - u = f(u)$, so (2) is formally the L^2 -gradient flow of E :

$$\partial_t u = -\nabla_{L^2} E(u) = \Delta u - (u^3 - u).$$

Remark 3.5 (Choice of energy). Antidifferentiating the nonlinearity $f = u^3 - u$ gives $F(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$, which yields the energy $\tilde{E}(u) = \int (\frac{1}{2}|\nabla u|^2 + F(u))$. One has $E = \tilde{E} + \frac{1}{4}|\Omega|$, so the two only differ by an additive constant and define the *same* gradient flow. We prefer the form (6) because (i) it is manifestly nonnegative, (ii) the double-well potential $W(u) = \frac{1}{4}(u^2 - 1)^2$ has its minima at $u = \pm 1$ visible to the naked eye, and (iii) it is the standard form used in the phase-transitions literature (Allen–Cahn, Ginzburg–Landau, Modica–Mortola).

Remark 3.6 (Dirichlet vs. other boundary conditions). This equation is usually studied with either homogeneous Dirichlet conditions ($u = 0$ on $\partial\Omega$, as here) or homogeneous Neumann conditions ($\partial_\nu u = 0$). The two cases are qualitatively close but have distinct equilibria (e.g. the constants $u \equiv \pm 1$ are stationary for Neumann but *not* for Dirichlet). The abstract theory applies to both; we restrict to Dirichlet to keep a single simple functional setting $V = H_0^1$. The variant with Dirichlet data is historically called the *Chafee–Infante* equation.

Remark 3.7 (Why $u_0 \in H_0^1$?). The choice of V as the state space is dictated by two consistent requirements. First, the nonlinearity $u \mapsto u^3$ is only controlled in L^2 via Sobolev, which in dimension $N \leq 3$ gives $V \hookrightarrow L^6$, and so $u^3 \in L^2$ for $u \in V$. Second, and more structurally, the energy (6) is finite exactly on V : the whole Lyapunov/LaSalle/Łojasiewicz–Simon story is driven by E , and L^2 initial data would have $E = +\infty$, making the programme vacuous.

4 Existence of the semigroup on V

4.1 A locally Lipschitz d operator

The nonlinearity $f(u) = u^3 - u$ is *not* Lipschitz from L^2 to L^2 (cubic growth). But it is Lipschitz on balls from V to L^2 , thanks to the Sobolev embedding $V \hookrightarrow L^6$.

Lemma 4.1 (Local Lipschitz property). *Define $\Phi : V \rightarrow L^2(\Omega)$, $\Phi(u) := u^3 - u$. Then Φ is locally Lipschitz: for every $R > 0$ there exists $L(R) > 0$ such that, for all $u, v \in V$ with $\|u\|_V, \|v\|_V \leq R$,*

$$\|\Phi(u) - \Phi(v)\|_{L^2} \leq L(R)\|u - v\|_V.$$

Moreover $\Phi(0) = 0$, so $\|\Phi(u)\|_{L^2} \leq L(R)\|u\|_V$ on $B_V(0, R)$.

Proof. Write $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$. Hölder with exponents $(6, 3)$ gives

$$\|u^3 - v^3\|_{L^2} \leq \|u - v\|_{L^6} \|u^2 + uv + v^2\|_{L^3}.$$

Since $\|u^2\|_{L^3} = \|u\|_{L^6}^2$ and $\|uv\|_{L^3} \leq \|u\|_{L^6}\|v\|_{L^6}$, we get $\|u^2 + uv + v^2\|_{L^3} \leq 3 \max(\|u\|_{L^6}, \|v\|_{L^6})^2 \leq 3C_S^2 R^2$. Together with $\|u - v\|_{L^6} \leq C_S\|u - v\|_V$ and $\|u - v\|_{L^2} \leq C_P\|u - v\|_V$: $\|\Phi(u) - \Phi(v)\|_{L^2} \leq (3C_S^3 R^2 + C_P)\|u - v\|_V$. $\square$

4.2 Local existence by Banach fixed point

A function $u \in C([0, T]; V)$ is a *mild solution* of (2) if

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\Phi(u(s)) ds, \quad t \in [0, T]. \quad (7)$$

Theorem 4.2 (Local existence). *For every $u_0 \in V$, there exist $T = T(\|u_0\|_V) > 0$ and a unique $u \in C([0, T]; V)$ satisfying (7).*

Proof. Set $R := 2\|u_0\|_V$, $L := L(R)$ from Lemma 4.1, and

$$\mathcal{X}_T := \{u \in C([0, T]; V) : \|u(t)\|_V \leq R \text{ for all } t \in [0, T]\},$$

a complete metric space for the sup- V norm. Define $\mathcal{N} : \mathcal{X}_T \rightarrow C([0, T]; V)$ by the right-hand side of (7).

Self-map. For $u \in \mathcal{X}_T$, Lemma 4.1 gives $\|\Phi(u(s))\|_{L^2} \leq LR$. By (5),

$$\left\| \int_0^t e^{-(t-s)A} \Phi(u(s)) ds \right\|_V \leq \int_0^t C(t-s)^{-1/2} LR ds = 2CLR\sqrt{t}.$$

Since $\|e^{-tA}u_0\|_V \leq \|u_0\|_V = R/2$, $\|\mathcal{N}(u)(t)\|_V \leq R/2 + 2CLR\sqrt{T}$. Choose T with $2CL\sqrt{T} \leq 1/2$; then $\mathcal{N}(u) \in \mathcal{X}_T$.

Contraction. For $u, v \in \mathcal{X}_T$,

$$\|\mathcal{N}(u)(t) - \mathcal{N}(v)(t)\|_V \leq \int_0^t C(t-s)^{-1/2} L \|u(s) - v(s)\|_V ds \leq 2CL\sqrt{T} \|u - v\|_{\mathcal{X}_T}.$$

With T as above, $\mathcal{N}$ is a strict contraction; Banach's theorem yields a unique mild solution. $\square$

4.3 The energy identity and global existence

Lemma 4.3 (Parabolic regularity, cited). *The mild solution u of Theorem 4.2 satisfies*

$$u \in C([0, T]; V) \cap L^2(0, T; D(A)) \cap H^1(0, T; L^2(\Omega)),$$

and $\partial_t u + Au = -\Phi(u)$ holds in $L^2((0, T) \times \Omega)$. Moreover, for every $\tau > 0$, $u \in C([\tau, T]; D(A))$ with $\|u(t)\|_{H^2}$ bounded by a constant depending only on τ and $\|u_0\|_V$.

This is classical analytic-semigroup theory; see [9, Ch. 3]. The key mechanism: e^{-tA} is an analytic semigroup on L^2 , so it has parabolic smoothing, and iteration in the Duhamel formula (7) propagates regularity.

Proposition 4.4 (Energy identity). *For every $0 \leq s \leq t \leq T$,*

$$E(u(t)) + \int_s^t \|\partial_\tau u(\tau)\|_{L^2}^2 d\tau = E(u(s)). \quad (8)$$

In particular, $t \mapsto E(u(t))$ is nonincreasing.

Proof. By Lemma 4.3, $u \in L^2(s, t; D(A))$ and $\partial_\tau u \in L^2(s, t; L^2)$. The standard chain-rule lemma (Lions–Magenes) gives $\frac{d}{d\tau} \frac{1}{2} \|\nabla u\|_{L^2}^2 = \langle -\Delta u, \partial_\tau u \rangle_{L^2}$. Similarly $\frac{d}{d\tau} \int W(u) = \langle W'(u), \partial_\tau u \rangle_{L^2} = \langle \Phi(u), \partial_\tau u \rangle_{L^2}$. Adding,

$$\frac{d}{d\tau} E(u) = \langle -\Delta u + \Phi(u), \partial_\tau u \rangle_{L^2} = \langle -\partial_\tau u, \partial_\tau u \rangle_{L^2} = -\|\partial_\tau u\|_{L^2}^2.$$

Integrating gives (8). $\square$

Corollary 4.5 (A priori V -bound). *Along any mild solution, $\frac{1}{2}\|u(t)\|_V^2 \leq E(u(t)) \leq E(u_0)$.*

Proof. $E(u) \geq \frac{1}{2}\|u\|_V^2$ since $W \geq 0$. Then use monotonicity. $\square$

Theorem 4.6 (Global semigroup). *For every $u_0 \in V$, the mild solution extends globally to $[0, \infty)$. The map $S_t u_0 := u(t)$ defines a continuous semigroup on V in the sense of Definition 1.1.*

Proof. Global existence. If $T_{\max} < \infty$, the blow-up alternative of the fixed-point argument gives $\limsup_{t \uparrow T_{\max}} \|u(t)\|_V = \infty$, contradicting Corollary 4.5.

Semigroup properties. (S1) is immediate from (7) at $t = 0$. (S2) follows from uniqueness: $t \mapsto u(t+s)$ and $t \mapsto S_t(u(s))$ are both mild solutions with initial datum $u(s)$. For (S3), a Gronwall argument on (7) using the local Lipschitz bound gives, on every interval $[0, T]$ where both orbits stay in a common V -ball, $\|S_t u_0 - S_t v_0\|_V \leq e^{CLt} \|u_0 - v_0\|_V$, which yields joint continuity. $\square$

5 LaSalle applied to Allen–Cahn

We now apply LaSalle’s principle (Theorem 3.1) to our concrete semigroup. The crucial task is to verify the precompactness hypothesis.

5.1 Precompactness of orbits

Lemma 5.1. *For every $u_0 \in V$, the orbit $\{u(t) : t \geq 1\}$ is relatively compact in V .*

Proof. By Corollary 4.5, the orbit is bounded in V . By Lemma 4.3 (with $\tau = 1$), $\|u(t)\|_{H^2}$ is bounded uniformly for $t \geq 1$ by a constant K depending only on $\|u_0\|_V$. Since the embedding $H^2 \cap H_0^1 \hookrightarrow V$ is compact (Rellich), the orbit $\{u(t) : t \geq 1\}$ is relatively compact in V . $\square$

5.2 The set of stationary solutions and asymptotic behaviour

Denote by $\mathcal{S}$ the set of stationary solutions of (2):

$$\mathcal{S} := \{\phi \in V : -\Delta\phi = \phi - \phi^3 \text{ in } \Omega, \phi|_{\partial\Omega} = 0\}.$$

Proposition 5.2 (LaSalle for Allen–Cahn). *For every $u_0 \in V$, the trajectory $u(t) = S_t u_0$ asymptotically approaches the set of stationary solutions:*

$$\text{dist}_V(u(t), \mathcal{S}) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Proof. We apply Theorem 3.1 with the metric space $X = V$, the semigroup $(S_t)_{t \geq 0}$ from Theorem 4.6, and the Lyapunov function E from (6), which is continuous on V (because $V \hookrightarrow L^4$), bounded below by 0, and nonincreasing along trajectories (Proposition 4.4). The relative compactness of $\gamma^+(u_0)$ in V for $t \geq 1$ is Lemma 5.1; for $t \in [0, 1]$ the orbit is compact in V by continuity. So Theorem 3.1 applies and gives $\text{dist}_V(u(t), \mathcal{I}) \rightarrow 0$, where $\mathcal{I}$ is the largest positively invariant subset of $\mathcal{E} = \{v \in V : E(S_t v) = E(v) \text{ for all } t \geq 0\}$.

It remains to identify $\mathcal{I}$ with $\mathcal{S}$. If $v \in \mathcal{I}$, then $v \in \mathcal{E}$, so $E(S_t v) = E(v)$ for all $t \geq 0$. The energy identity (8) applied to the trajectory $\tau \mapsto S_\tau v$ then gives

$$\int_0^t \|\partial_\tau S_\tau v\|_{L^2}^2 d\tau = E(v) - E(S_t v) = 0 \quad \forall t \geq 0,$$

so $\partial_\tau S_\tau v \equiv 0$, i.e. $S_\tau v = v$ for all $\tau \geq 0$. The PDE then reads $-\Delta v + v^3 - v = 0$, so $v \in \mathcal{S}$. Conversely, every $\phi \in \mathcal{S}$ is a fixed point of (S_t) , so $\{\phi\}$ is positively invariant and contained in $\mathcal{E}$, hence in $\mathcal{I}$ by maximality. Thus $\mathcal{I} = \mathcal{S}$ and the conclusion follows. $\square$

Remark 5.3 (Why this is already a remarkable conclusion). A priori, the long-time behaviour of a semilinear parabolic PDE could be very wild. Proposition 5.2 says that, whatever the initial datum, the solution asymptotically gets arbitrarily close to the set of solutions of a *nonlinear elliptic equation*. The infinite-dimensional parabolic dynamics is thus entirely driven by the structure of the stationary set $\mathcal{S}$.

Remark 5.4 (Stationary set for Chafee–Infante). For $\Omega = (0, L)$ a one-dimensional interval, the set $\mathcal{S}$ undergoes a classical bifurcation cascade as L crosses the thresholds $k\pi$, $k \geq 1$: each crossing creates a new pair of nontrivial equilibria branching off the trivial solution 0. Already for $L > \pi$, the trivial equilibrium becomes unstable and two new stable equilibria appear. See Chafee–Infante [4] or [9, Ch. 5].

6 Convergence to a single equilibrium: Łojasiewicz–Simon

Proposition 5.2 tells us $\text{dist}_V(u(t), \mathcal{S}) \rightarrow 0$, but a priori the trajectory could “wander” along $\mathcal{S}$ if the latter contains a continuum of equilibria. In finite dimensions, this is prevented by the classical Łojasiewicz inequality for analytic functions; in infinite dimensions, the corresponding tool is Simon’s extension.

6.1 The Łojasiewicz–Simon inequality

The energy E in (6) is real-analytic on V because the potential $W(u) = \frac{1}{4}(u^2 - 1)^2$ is polynomial. Its Fréchet derivative is

$$\langle E'(v), \varphi \rangle_{V', V} = \int_{\Omega} (\nabla v \cdot \nabla \varphi + (v^3 - v)\varphi) dx,$$

which means $E'(v) = -\Delta v + v^3 - v \in H^{-1}(\Omega)$. Critical points of E are exactly the elements of $\mathcal{S}$.

Theorem 6.1 (Łojasiewicz–Simon inequality). *Let $\phi^* \in \mathcal{S}$. There exist $\theta \in (0, \frac{1}{2}]$, $\sigma > 0$, and $c > 0$ such that, for every $v \in V$ with $\|v - \phi^*\|_V \leq \sigma$,*

$$|E(v) - E(\phi^*)|^{1-\theta} \leq c \|E'(v)\|_{H^{-1}}. \quad (9)$$

We take (9) as a black box. The strategy of proof is as follows. Near ϕ^* , the Hessian $E''(\phi^*)\varphi = -\Delta\varphi + (3(\phi^*)^2 - 1)\varphi$ is a compact perturbation of $-\Delta$, hence a Fredholm operator of index zero. A Lyapunov–Schmidt reduction splits V into a finite-dimensional kernel N and an infinite-dimensional part on which $E''(\phi^*)$ restricts to an isomorphism on $N^\perp$. On the invertible part, a quadratic Taylor expansion gives (9) trivially with $\theta = \frac{1}{2}$. On N , the analytic structure reduces the problem to a *finite-dimensional* Łojasiewicz inequality for analytic functions, which holds by Łojasiewicz’s classical theorem. Complete proofs are in Simon [12] (the original), the monograph of Haraux–Jendoubi [8], and Chill [5].

Remark 6.2 (Geometric meaning). In finite dimensions, the classical Łojasiewicz inequality says that near a critical point of an analytic function, the gradient controls a positive power of the oscillation. In dimension one, if $e(s) = s^{2k}$, one checks directly $\theta = 1/(2k)$. The exponent θ measures how “degenerate” the critical point is: $\theta = 1/2$ for non-degenerate critical points (exponential rate of convergence); $\theta < 1/2$ otherwise (algebraic rate).

Remark 6.3 (Why analyticity is essential). The conclusion of Theorem 6.1 is *false* in general for merely C^∞ energies. There exist smooth gradient systems in the plane with trajectories accumulating on an entire circle (without convergence). The polynomial nature of W is therefore not a mere convenience: it is what drives single-equilibrium convergence.

6.2 Convergence to a single equilibrium

We can now conclude the main theorem. The argument is known as the *Łojasiewicz–Simon method*; we follow the presentation of [8, Ch. 3].

Theorem 6.4 (Convergence). *For every $u_0 \in V$, there exists $\phi_\infty \in \mathcal{S}$ such that*

$$u(t) = S_t u_0 \xrightarrow[t \rightarrow \infty]{V} \phi_\infty.$$

Proof. Let $E_\infty := \lim_{t \rightarrow \infty} E(u(t))$, which exists by monotonicity of E along the orbit. By Proposition 5.2 and the relative compactness of the orbit (Lemma 5.1), there exists a sequence $t_n \rightarrow \infty$ and $\phi^* \in \mathcal{S}$ such that $u(t_n) \rightarrow \phi^*$ in V ; by continuity of E , $E(\phi^*) = E_\infty$. We show $u(t) \rightarrow \phi^*$ in V .

Case 1: $E(u(t_0)) = E_\infty$ for some t_0 . By monotonicity, $E(u(t)) = E_\infty$ for all $t \geq t_0$; the energy identity then gives $\partial_t u \equiv 0$ on $[t_0, \infty)$, so $u(t) = u(t_0) \in \mathcal{S}$ and the conclusion is trivial.

Case 2: $E(u(t)) > E_\infty$ for all t . We assume $t \geq 1$ so that $\|u(t)\|_{H^2} \leq K$ (Lemma 4.3). Apply Theorem 6.1 at ϕ^* , with constants θ, σ, c . Define

$$H(t) := (E(u(t)) - E_\infty)^\theta > 0.$$

H is C^1 as long as $E(u(t)) > E_\infty$. Using Proposition 4.4,

$$-\frac{d}{dt}H(t) = -\theta(E(u(t)) - E_\infty)^{\theta-1} \frac{d}{dt}E(u(t)) = \theta \frac{\|\partial_t u(t)\|_{L^2}^2}{(E(u(t)) - E_\infty)^{1-\theta}}.$$

Now the PDE reads $\partial_t u = -E'(u)$ in L^2 , hence in H^{-1} , so $\|E'(u(t))\|_{H^{-1}} = \|\partial_t u(t)\|_{H^{-1}} \leq C_P \|\partial_t u(t)\|_{L^2}$ by the continuous embedding $L^2 \hookrightarrow H^{-1}$. Whenever $u(t) \in B_V(\phi^*, \sigma)$, Theorem 6.1 gives $(E(u(t)) - E_\infty)^{1-\theta} \leq cC_P \|\partial_t u(t)\|_{L^2}$, so (assuming $\partial_t u(t) \neq 0$; else we are in Case 1 after time t)

$$-\frac{d}{dt}H(t) \geq \frac{\theta}{cC_P} \|\partial_t u(t)\|_{L^2}. \quad (10)$$

Integrating on any interval $[t_1, t_2]$ over which $u(t)$ remains in $B_V(\phi^*, \sigma)$:

$$\int_{t_1}^{t_2} \|\partial_\tau u\|_{L^2} d\tau \leq \frac{cC_P}{\theta} (H(t_1) - H(t_2)) \leq \frac{cC_P}{\theta} H(t_1). \quad (11)$$

Trapping the orbit near ϕ^* . Recall $u(t_n) \rightarrow \phi^*$ in V for some sequence $t_n \rightarrow \infty$; hence $H(t_n) \rightarrow 0$ by continuity of E . Fix n large enough that $t_n \geq 1$,

$$\|u(t_n) - \phi^*\|_V < \sigma/2, \quad (2K)^{1/2} \left(\frac{cC_P}{\theta} H(t_n) \right)^{1/2} < \sigma/2.$$

Set $\tau^* := \sup\{\tau \geq t_n : u(s) \in B_V(\phi^*, \sigma) \text{ for all } s \in [t_n, \tau]\}$. For $s \in [t_n, \tau^*]$, (11) applies on $[t_n, s]$, so

$$\|u(s) - u(t_n)\|_{L^2} \leq \int_{t_n}^s \|\partial_\tau u\|_{L^2} d\tau \leq \frac{cC_P}{\theta} H(t_n).$$

We now use the *interpolation*

$$\|w\|_V^2 = \int |\nabla w|^2 = - \int w \Delta w \leq \|w\|_{L^2} \|w\|_{H^2}, \quad (12)$$

valid for $w \in H^2 \cap H_0^1$. Applied to $w = u(s) - u(t_n) \in H^2 \cap H_0^1$ with $\|w\|_{H^2} \leq 2K$ (Lemma 4.3),

$$\|u(s) - u(t_n)\|_V \leq (2K)^{1/2} \|u(s) - u(t_n)\|_{L^2}^{1/2} \leq (2K)^{1/2} \left(\frac{cC_P}{\theta} H(t_n) \right)^{1/2} < \sigma/2.$$

Combined with $\|u(t_n) - \phi^*\|_V < \sigma/2$, this yields $\|u(s) - \phi^*\|_V < \sigma$. By continuity of $s \mapsto u(s)$, this strict inequality rules out $\tau^* < \infty$; hence $\tau^* = \infty$.

Convergence. The estimate (11) now applies on $[t_n, \infty)$:

$$\int_{t_n}^\infty \|\partial_\tau u\|_{L^2} d\tau \leq \frac{cC_P}{\theta} H(t_n) < \infty.$$

Hence $u(\cdot)$ is Cauchy in L^2 as $t \rightarrow \infty$: for any $s < t$ large, $\|u(t) - u(s)\|_{L^2} \leq \int_s^t \|\partial_\tau u\|_{L^2} \rightarrow 0$. By the interpolation (12) and the uniform H^2 -bound, $u(\cdot)$ is also Cauchy in V , so $u(t) \rightarrow \phi_\infty$ in V for some $\phi_\infty \in V$. By the trapping argument above, $\phi_\infty = \phi^* \in \mathcal{S}$. $\square$

Remark 6.5. Reading back the three main steps:

- Fixed point (Theorem 4.6): *the PDE defines a dynamics.*
- LaSalle (Proposition 5.2): *the dynamics asymptotically concentrates on stationary solutions.*
- Łojasiewicz–Simon (Theorem 6.4): *it actually selects a single one.*

Each step is necessary: LaSalle alone would allow a continuum of accumulation points, and a finite-dimensional Łojasiewicz argument cannot apply because V is infinite-dimensional. It is the combination of (i) parabolic compactness, (ii) nonstrict Lyapunov dissipation, and (iii) analyticity of the nonlinearity, which yields this structural rigidity.

Going further

Beyond the specific equation treated here, many directions open up:

- **Genuinely nonlinear semigroups.** The porous medium equation, the p -Laplacian evolution, and many others cannot be treated by a fixed-point argument of the type used here: the operator is non-Lipschitz and even multivalued. The right tool is the Crandall–Liggett theorem for m -accretive operators, see [6, 2].
- **Gradient flows in metric spaces.** When the ambient space is a genuine metric space (e.g. the Wasserstein space of probability measures, or metric spaces of nonpositive curvature), gradient flows can still be defined via curves of maximal slope; see Ambrosio–Gigli–Savaré [1].
- **Dynamical systems viewpoint.** Attractors, invariant manifolds and bifurcations for semilinear parabolic equations are developed in Henry [9] and Haraux [7].
- **Convergence and rates via Łojasiewicz–Simon.** Many dissipative PDEs admit a Łojasiewicz–Simon inequality and hence single-equilibrium convergence. The monograph [8] surveys the state of the art, including explicit convergence rates.
- **Control and stabilization.** LaSalle’s principle plays a central role in many infinite-dimensional feedback stabilization results. The original monograph is [10].

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